Quantization of affine function spaces

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Joint work with

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Preliminaries

- Let K be a compact convex set in a locally convex space E.
- $a : K \to \mathbb{R}$ is affine map if

$$a(\lambda u + (1-\lambda)v) = \lambda a(u) + (1-\lambda)a(v)$$

for all $u, v \in K$ and $\lambda \in [0,1]$.

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- $A(K)$: the set of all continuous affine functions on K.

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$$\|x^*x\| = \|x\|^2 \qquad \forall x \in \mathcal{A}.$$

- $\mathcal{A}_{sa}$: the set of all self-adjoint element of $\mathcal{A}$
- $\mathcal{A}^+$: the set of all positive elements of $\mathcal{A}$ (i.e. $a \in \mathcal{A}^+$ if $a = b^*b$ for some $b \in \mathcal{A}$).

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Kadison's observation

Let $\mathcal{B} \subset \mathcal{A}$ be a unital self-adjoint subspace. Let $\mathcal{B}^+ = \mathcal{B} \cap \mathcal{A}^+$.

Then $(\mathcal{B}_{sa}, \mathcal{B}^+, I)$ is an **order unit space**. That is

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1. (order unit property) $b \in B \implies -\lambda I \leq b \leq \lambda I$ for some $\lambda > 0$.
2. (Archimedean property) if $b + \lambda I \geq 0$ for all $\lambda > 0$ for some $b \in A \implies b \geq 0$.

Affine function space and order unit space

- A linear function $f : \mathcal{B} \to \mathbb{C}$ is called **state** if

$$f(\mathcal{B}^+) \subseteq [0, \infty) \quad \text{and} \quad f(1) = 1.$$

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Problem: Is it possible to construct a **compact convex set** K such that

$$A(K) \cong \mathcal{B}_{sa},$$

where $\mathcal{B}$ is a unital self-adjoint subspace of a C^*-algebra.

Non-comutaive order, Effros-1976

• Non self-adjoint characterization : Let $a \in \mathcal{B}$. Then $\|a\| \leq 1$ if and only if

$$\begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix} \in M_2(\mathcal{B})^+.$$

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- If $a \in \mathcal{B}$, then

$$\|a\| = \inf\{\lambda \geq 0 : \begin{bmatrix} \lambda I & a \\ a^* & \lambda I \end{bmatrix} \geq 0\}.$$

Non commutative ordered in abstract space

Definition (Choi, Effros-77)

Let V be a complex $*$-vector space. Then V is called **matrix ordered space** if there is a cone $M_n(V)^+ \subset M_n(V)_{sa}$ for each n such that

$$\gamma^* M_m(V)^+ \gamma \subset M_n(V)^+$$

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- $\phi : V \to W$ be a linear map (V, W vector spaces).
- (*n*-**amplification**) $\phi_n : M_n(V) \to M_n(W)$ by

$$\phi_n([v_{i,j}]) = [\phi(v_{i,j})].$$

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- Let V, W be matrix ordered spaces and $\phi : V \to W$ be a self-adjoint linear map. Then ϕ is **completely positive** if ϕ_n is positive for each n.

Theorem (Choi-Effros–1977)

Let $(V, \{M_n(V)^+\})$ be a matrix ordered space and let e be an order unit for V^+. Then following are equivalent

1. V^+ *is* **proper** *and* $M_n(V)^+$ **Archimedean** *for each* n
2. *there exists* **cp-map** $\varphi : V \to \mathcal{B}(H)$ *for some Hilbert space* H.

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Let V be a complex vector space, and let $\|.\|_n$ be a norm on $M_n(V)$. Then V is called a **matrcially normed space** if

1. $\|\alpha v \beta\|_n \leq \|\alpha\| \|v\|_n \|\beta\|$ for all $v \in M_n(V), \alpha, \beta \in \mathbb{M}_n$,
2. $\|v \oplus 0\|_{n+m} = \|v\|_n$ for all $v \in M_n(V)$.

Choi, Effros, *Injectivity of operator spaces*, J. Funct. Anal. **24** (1977), 156–209.

L^{∞}-matricial norm space

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Theorem (Ruan–1988)

Let $(V, \{\|\cdot\|_n\})$ be a matricially normed space. Then V is an L^∞-matrcially normed space $\iff$ there is a **complete isometry** *$\phi : V \to \mathcal{B}(H)$ for some Hilbert space H.*

Ruan, *Subspaces of C*-algebras*, J. Funct. Anal., **98** (1988), 217–230.

C^*-ordered operator space

Definition

Let $(V, \{M_n(V)^+\})$ be a matrix ordered space together with an L^∞-matricial norm $\{\|\cdot\|_n\}$ is said to be a C^*-**ordered operator space** if V^+ is proper and for each $n \in \mathbb{N}$, satisfies the following:

1. $*$ is isometry on $M_n(V)$;
2. $M_n(V)^+$ is **closed**;
3. $\|f\|_n \leq \max\{\|g\|_n, \|h\|_n\}$, whenever $f \leq g \leq h$ with $f, g, h \in M_n(V)_{sa}$.

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Theorem (Karn, 2011)

Let $(V, \{M_n(V)^+\})$ be a matrix ordered space with an L^∞-matricial norm $\{\|\cdot\|_n\}$.
Then V is a C^*-**ordered operator space** $\iff$ *there exists a* **completely order isomerty** $\varphi : V \to \mathcal{A}$ *for some C^*-algebra $\mathcal{A}$.*

A. K. Karn, Order embedding of matrix ordered spaces, Bull. Aust. Math. Soc. 84 (2011), 10–18.

Geometry of convex set

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Definition (G.-Karn)

An element $k \in K$ is called a **lead point** of K ($k \in \text{lead}(K)$) if $k = \alpha k_1$ for some $k_1 \in K$ with $\alpha \in [0, 1]$, then $\alpha = 1$.

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Theorem (G.-Karn)

For each $k \in K \setminus \{0\}$. There is a **unique** $\alpha \in (0,1]$ *and* $\widehat{k} \in \text{lead}(K)$ *such that* $k = \alpha \widehat{k}$.

Quantization

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- (Recall) $K_n \in M_n(V)_{sa}$ and $0 \in \text{ext}\{K_n\}$.

• **Reformulation of the problem:** Is it possible to find a sequence $\{K_n\}$ of **compact convex** set so that
the affine spaces over K_n is turn out to a C^*-ordered operator space ?

L^1-matrix convex set

L^1-matrix convex set (G.-Karn)

Let V be a $*$-locally convex space. Let $\{K_n\}$ be a collection of compact convex sets with $K_n \subset M_n(V)_{sa}$ and $0 \in \partial_e(K_n)$ for all n. Then $\{K_n\}$ is called an L^1-**matrix convex set** if the following conditions hold:

L_1 If $u \in K_n$ and $\gamma_i \in \mathbb{M}_{n,n_i}$ such that $\sum_{i=1}^k \gamma_i \gamma_i^* \leq I_n$, then $\oplus_{i=1}^k \gamma_i^* u \gamma_i \in K_{\sum_{i=1}^k n_i}$.

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L_2 If $u \in K_{2n}$ so that $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^* & u_{22} \end{bmatrix}$ for some $u_{11}, u_{22} \in K_n$ and $u_{12} \in M_n(V)$, then $u_{12} + u_{12}^* \in \mathrm{co}(K_n \cup -K_n)$.

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L₃ Let $u \in K_{m+n}$ with $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^* & u_{22} \end{bmatrix}$ so that $u_{11} \in K_m, u_{22} \in K_n$ and $u_{12} \in M_{m,n}(V)$ and if $u_{11} = \alpha_1 \widehat{u_{11}}, u_{22} = \alpha_{22} \widehat{u_{22}}$ with $\widehat{u_{11}} \in \mathrm{lead}(K_m), \widehat{u_{22}} \in \mathrm{lead}(K_n)$, then $\alpha_1 + \alpha_2 \leq 1$.

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•**Example:** Let V be a C^*-ordered operator space. Then $\{Q_n(V)\}$ is an L^1-matrix convex set with $\text{lead}(Q_n(V)) = S_n(V)$.

$\{A_0(K_n)\}$-Spaces

- Let V be a $*$-locally convex space,
- let $\{K_n\}$ be an L^1-matrix convex set of V,
- $M_n(V)^+ := \cup_{r=1}^{\infty} rK_n$ is a cone in $M_n(V)_{sa}$ for all n
- with V^+ is proper and generating.

Define

$$A_0(K_n, M_n(V)) := \{a : K_n \to \mathbb{C} \mid a \text{ is continuous and affine;}$$
$$a(0) = 0; \text{and } a \text{ extends to a continuous linear functional}$$
$$\tilde{a} : M_n(V) \to \mathbb{C}\}.$$

Main Theorem

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Theorem (G.-Karn)

Let $\{K_n\}$ *be an* L^1*-matrix convex set of* V.

$\Longrightarrow$ $(A_0(K_1, V), \{M_n(A_0(K_1, V)^+\}, \{\|.\|_n\})$ *is a* C^**-ordered operator space.*

Sketch of the proof:

Step-1 Given $a \in A_0(K_n, M_n(V))$, we define

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Step-2 $A_0(K_n, M_n(V))_{sa} = \{a \in A_0(K_n, M_n(V)) : a^* = a\}$.

Continued..

Step-3 If $\alpha \in \mathbb{M}_{m,n}$, $\beta \in \mathbb{M}_{n,m}$ and $a \in A_0(K_n, M_n(V))$. We define

$$\alpha a \beta(v) := \tilde{a}(\alpha^T v \beta^T) \text{ for all } v \in K_m.$$

- Then $\alpha a \beta \in A_0(K_m, M_m(V))$

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Step-4 If $a \in A_0(K_n, M_n(V))$ and $b \in A_0(K_m, M_m(V))$. We define

$$(a \oplus b)(v) := a(v_{11}) + b(v_{22}) \text{ for all } v \in K_{n+m},$$

where $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix}$ for some $v_{11} \in K_n, v_{22} \in K_m, v_{12} \in M_{n,m}(V)$.

- Then $a \oplus b \in A_0(K_{n+m}, M_{n+m}(V))$.

Continued..

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$$A_0(K_n, M_n(V))^+ := \{a \in A_0(K_n, M_n(V))_{sa} : a(u) \geq 0 \ \ \forall u \in K_n\}.$$

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Step-6 $A_0(K_n, M_n(V))$ is a normed linear space equipped with norm

$$\|a\|_{\infty,n} = \sup\left\{ \left| \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} (u) \right| : u \in K_{\mathbf{2n}} \right\} \text{ for all } a \in A_0(K_n, M_n(V)).$$

Continued..

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Step-6 $A_0(K_n, M_n(V))$ is a normed linear space equipped with norm

$$\|a\|_{\infty,n} = \sup\left\{\left|\begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix}(u)\right| : u \in K_{\mathbf{2n}}\right\} \text{ for all } a \in A_0(K_n, M_n(V)).$$

Step-7
If $a \in A_0(K_n, M_n(V))_{sa}$, then we have

$$\|a\|_{\infty,n} = \sup\{|a(v)| : v \in K_n\}.$$

Continued..

Step-8 If $a \in A_0(K_n, M_n(V))$.

$$\|a^*\|_{\infty,n} = \|a\|_{\infty,n}.$$

Continued..

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Step-9 $\{\|\cdot\|_{\infty,n}\}$ satisfies the following conditions:

1. If $a \in A_0(K_m, M_m(V)), b \in A_0(K_n, M_n(V))$; then

$$\|a \oplus b\|_{\infty,m+n} = \max\{\|a\|_{\infty,m}, \|b\|_{\infty,n}\}.$$

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2. if $a \in A_0(K_m, M_n(V)), \alpha \in \mathbb{M}_{m,n}$ and $\beta \in \mathbb{M}_{n,m}$;
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$$\|\alpha a \beta\|_{\infty,m} \leq \|\alpha\|\|a\|_{\infty,n}\|\beta\|$$

3. If $a \leq b \leq c$ in $A_0(K_n, M_n(V))_{sa}$, we have
$$\|b\|_{\infty,n} \leq \max\{\|a\|_{\infty,n}, \|c\|_{\infty,n}\}.$$

Step-10 Let us define a map
$\Phi_n : A_0(K_n, M_n(V)) \mapsto M_n(A_0(K_1, V))$ given by

$$\phi_n(a) = [a_{i,j}], \tag{1}$$

where $a_{i,j}(v) = \tilde{a}(\varepsilon_{i,j} \otimes v)$ for all $v \in K_1$.

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$\Phi_n : A_0(K_n, M_n(V)) \mapsto M_n(A_0(K_1, V))$ is a $*$-isomorphism.

Step -12 We transport the **order** and the **norm structures** to $M_n(A_0(K_1, V))$ via isomorphism from $A_0(K_n, M_n(V))$ as follows:

$$[a_{i,j}] \in M_n(A_0(K_1, V))^+$$

if and only if

$$\Sigma_{i,j=1}^{n} \tilde{a_{i,j}}(v_{i,j}) \geq 0 \qquad (2)$$

for all $[v_{i,j}] \in K_n$ and

$$\|[a_{i,j}]\|_n = \|\Phi_n^{-1}([a_{i,j}])\|_{\infty,n} \text{ for all } [a_{i,j}] \in M_n(A_0(K_1, V)). \qquad (3)$$

Step-13 Now $(A_0(K_1, V), \{M_n(A_0(K_1, V))^+\}, \{\|.\|_n\})$ is a **matrix ordered** and L^∞-**matricially normed space** such that

1. $*$ is an isometry,
2. If $[a_{i,j}], [b_{i,j}], [c_{i,j}] \in M_n(A_0(K_1, V))_{sa}$ with

$$[a_{i,j}] \leq [b_{i,j}] \leq [c_{i,j}],$$

   we have

$$\|[b_{i,j}]\|_n \leq \max\{\|[a_{i,j}]\|_n, \|[c_{i,j}]\|_n\}.$$

Hence, $\{A_0(K_n, M_n(V))\}$ is **complete isometrically, completely order isomorphic** to a C^*-ordered operator space.

L^1-regularly embedded (G.- Karn)

Let $\{K_n\}$ be an L^1-matrix convex set in a $*$-locally convex space V. Then $\{K_n\}$ is called L^1-**regularly embedded** in V if $\mathbf{L_1}(= \text{lead}(K_1))$ is regularly embedded in V_{sa}. In other words,

1. L_1 is compact and convex; and
2. $\chi : V_{sa} \to (A(L_1)^*_{sa})_{w*}$ is a linear homeomorphism.

Here $\chi(w)(a) = \lambda a(u) - \mu a(v)$ for all $a \in A(L_1)_{sa}$ if $w = \lambda u - \mu v$ for some $u, v \in L_1$ and $\lambda, \mu \in \mathbb{R}^+$.

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L^1-matricial cap (G.- Karn)

Let $\{K_n\}$ be an L^1-matrix convex set. Then $\{L_n\}$ is called L^1-*matricial cap* for V if

1. L_1 is convex and
2. if $v \in L_{m+n}$ with $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix}$ for some $v_{11} \in K_m, v_{22} \in K_n$ and $v_{12} \in M_{m,n}(V)$ so that $v_{11} = \alpha_1 \widehat{v_1}, v_{22} = \alpha_{22} \widehat{v_2}$ for some $\widehat{v_1} \in L_m, \widehat{v_2} \in L_n$ and $\alpha_1, \alpha_2 \in [0,1]$, then $\alpha_1 + \alpha_2 = 1$.

Characterization of operator system

Theorem (G.- Karn)

Let $\{K_n\}$ be an L^1-**regularly embedded** in V such that $\{L_n\}$ is an L^1-**matricial cap** for V. Then

1. $A_0(K_n, M_n(V))$ is an **order unit space** for all $n \in \mathbb{N}$.
2. $\{A_0(K_1, V), \{M_n(A_0(K_1, V)\}, e)\}$ is an **abstract operator system**.

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THANK YOU!